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POINTWISE CONVERGENCE OF MARTINGALES IN VON NEUMANN ALGEBRAS

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ABSTRACT
We prove the pointwise convergence of martingales in a von Neumann algebra.

I. Introduction

Recently E. C. Lance proved the pointwise ergodic theorem for actions of the group of integers on von Neumann algebras (cf. [4]). In a joint work with J.-P. Conze (cf. [1]), we have extended other pointwise ergodic theorems to von Neumann algebra context. The purpose of this paper is to extend another convergence theorem, the pointwise convergence theorem for martingales, to von Neumann algebras. Our proof uses E. C. Lance's results and a technical lemma of J. Neveu.

We fix a pair $(\mathscr{A}, \rho)$ where $\mathscr{A}$ is a von Neumann algebra and ρ is a faithful normal state on $\mathscr{A}$. As in the commutative case, we call *kernel* a positive linear contraction T of $\mathscr{A}$ into itself such that $T1 = 1$, $\rho(Ta) = \rho(a)$, $\rho((Ta)^*Ta) \leqq \rho(a^*a)$, $\forall a \in \mathscr{A}$. Let $\mathfrak{H}$ be the Hilbert space of the cyclic representation of $\mathscr{A}$ associated to ρ, ξ the cyclic and separating vector. If T is a kernel, we associate to T a canonical contraction $\tilde{T}$ of $\mathfrak{H}$ defined by $\tilde{T}(a\xi) = (Ta)\xi$, $\forall a \in \mathscr{A}$.

Following E. C. Lance (cf. [4]) we say that a sequence $(a_n)_{n \geqq 1}$ of $\mathscr{A}$ *converges* ρ*-almost uniformly* to an element a of $\mathscr{A}$ if for every $\varepsilon > 0$, there exists a projection e of $\mathscr{A}$ such that $\rho(e) \geqq 1 - \varepsilon$ and $\lim_n \|(a - a_n)e\| = 0$ (when $\mathscr{A}$ is commutative, $\mathscr{A} = L^\infty(X, \mu)$, with (X, μ) a probability space, this convergence coincides, via Egorov's theorem, with the almost everywhere pointwise convergence).

If $a \in \mathscr{A}$, by expressing a in terms of its self adjoint and skew-adjoint parts

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and then each of these as the difference of its positive and negative parts, we obtain

$$a=\sum_{j=1}^{4} i^{j-1}a_{(j)}, \qquad \text{with } a_{(j)}\in \mathcal{A}_+, \quad j=1,2,3,4.$$

We define $a^{++}=\sum_{j=1}^4 a_{(j)}$.

We recall the following results of E. C. Lance (cf. [4]).

LEMMA 1. (1) *Let a and b be bounded operators in* $\mathfrak{H}$ *satisfying* $0\leqq a\leqq b\leqq 1$ *and e a projection, then* $\|ae\|\leqq\|be\|^{1/2}$.

(2) *If* $\{b_k\}_{k\geqq 1}$ *is a bounded sequence in* $\mathcal{A}$ *which converges to zero ultrastrongly, then so is* (b_k^{++}).

(3) *Let T be a kernel on* $\mathcal{A}$. *For every* $a\in\mathcal{A}_+$, *there exists* $c\in\mathcal{A}_+$ *such that* $\|c\|\leqq 2\|a\|$, $\rho(c)\leqq 4\|a\|^{1/2}\rho(a)^{1/2}$ *and* $(1/n)\sum_0^{n-1}T^k a\leqq c$, *for every* $n\geqq 1$. *Furthermore for every* $a\in\mathcal{A}$ $(1/n)\sum_0^{n-1}T^k a$ *converges* ρ*-almost uniformly to a* T*-invariant element* a_T.

Following M. Takesaki (cf. [8]), we call *conditional expectation* of $\mathcal{A}$ onto a sub von Neumann algebra $\mathcal{B}$ (with respect to ρ), an idempotent kernel $E^{\mathcal{B}}$ of $\mathcal{A}$ onto $\mathcal{B}$ such that $E^{\mathcal{B}}(bab')=bE^{\mathcal{B}}ab'$, $b,b'\in\mathcal{B}$, $\forall a\in\mathcal{A}$. M. Takesaki has shown that the conditional expectation $E^{\mathcal{B}}$ of $\mathcal{A}$ onto $\mathcal{B}$ exists if and only if $\mathcal{B}$ is globally invariant by the modular automorphisms σ_t of the state ρ (we always have the uniqueness of the conditional expectation).

If $\{\mathcal{A}_n\}$ is a sequence of sub von Neumann algebras of $\mathcal{A}$, we note $\bigvee_{n\geqq 1}\mathcal{A}_n$ the von Neumann algebra generated by the $\mathcal{A}_n$'s. By Takesaki's result, if each $\mathcal{A}_n$ possesses a conditional expectation then so do $\bigvee_{n\geqq 1}\mathcal{A}_n$ and $\bigcap_{n\geqq 1}\mathcal{A}_n$.

To simplify the notations, we write $E_n=E^{\mathcal{A}_n}$.

For the results concerning classical martingale theory see [3], for other notations concerning von Neumann algebras see [2].

II. Convergence theorem for decreasing sequences of conditional expectations

We recall the following technical result of J. Neveu (cf. [5]) with its proof.

LEMMA 2. *Let* $\{\mathcal{A}_n\}$ *be a decreasing sequence of sub von Neumann algebras of* $\mathcal{A}$ *with conditional expectations. The operator*

$$T=\sum_{n\geqq 1}(a_{n+1}-a_n)E_n$$

is a kernel of $\mathscr{A}$ *if the sequence* (a_n) *of scalars satisfies* $a_1 = 0 < a_2 < \cdots < a_n < \cdots < 1$, $a_n \nearrow 1$.

Furthermore for every $\varepsilon > 0$, *we can choose the sequence* (a_n) *and a sequence* (q_n) *of integers such that*

$$\sum_{n \geqq 1} \left\| \frac{1}{q_n} \sum_{0 \leqq k < q_n} T^k - E_n \right\| \leqq \varepsilon.$$

PROOF. (1) Let $\tilde{E}_n$ be the contraction of $\mathfrak{H}$ corresponding to the kernel E_n: $E_n(a\xi) = (E_n a)\xi$, $\forall a \in \mathscr{A}$. It is clear that T is a positive linear contraction of $\mathscr{A}$, $T1 = 1$, $\rho(Ta) = \rho(a)$, $\forall a \in \mathscr{A}$; let

$$\tilde{T} = \sum_{n \geqq 1} (a_{n+1} - a_n)\tilde{E}_n,$$

then $\tilde{T}$ is a linear contraction of $\mathfrak{H}$ and $(Ta)\xi = \tilde{T}(a\xi)$, $\forall a \in \mathscr{A}$, therefore

$$\rho(Ta^*Ta) = \langle Ta\xi, Ta\xi \rangle = \| Ta\xi \|^2 \leqq \| a\xi \|^2 = \rho(a^*a).$$

We have proved that T is a kernel.

(2) By the formula $E_n E_m = E_m E_n = E_m$ if $m \geqq n$; we deduce that $T^k = \sum_{p \geqq 1}(a_{p+1}^k - a_p^k)E_p$; $\forall k \geqq 1$ and

$$\frac{1}{q} \sum_{0 \leqq k < q} T^k = \frac{1}{q} I + \sum_{p \geqq 1} \left(\frac{1 - a_{p+1}^q}{q(1 - a_{p+1})} - \frac{1 - a_p^q}{q(1 - a_p)} \right) E_p,$$

$$\left\| \frac{1}{q} \sum_{0 \leqq k < q} T^k - E_n \right\| \leqq \frac{1}{q} + \sum_{1 \leqq p \neq n} \left(\frac{1 - a_{p+1}^q}{q(1 - a_{p+1})} - \frac{1 - a_p^q}{q(1 - a_p)} \right) + \left(1 - \frac{1 - a_{n+1}^q}{q(1 - a_{n+1})} + \frac{1 - a_n^q}{q(1 - a_n)} \right),$$

$$\left\| \frac{1}{q} \sum_{0 \leqq k < q} T^k - E_n \right\| \leqq 2 \left(1 - \frac{1 - a_{n+1}^q}{q(1 - a_{n+1})} + \frac{1 - a_n^q}{q(1 - a_n)} \right).$$

If we take $q = q_n$ with

$$q_n - 1 < [(1 - a_n)(1 - a_{n+1})]^{-1/2} \leqq q_n,$$

we have

$$\left\| \frac{1}{q_n} \sum_{0 \leqq k < q_n} T^k - E_n \right\| \leqq 3 \left(\frac{1 - a_{n+1}}{1 - a_n} \right)^{1/2}.$$

It suffices then to choose the a_n's such that

$$\sum_{n \geqq 1} \left(\frac{1 - a_{n+1}}{1 - a_n} \right)^{1/2} \leqq \frac{\varepsilon}{3}$$

(in particular we can take $a_n = 1 - (1 + 3/\varepsilon)^{-2n^2}$). q.e.d.

PROPOSITION 3. *Let $\{\mathcal{A}_n\}$ be an increasing (resp. decreasing) sequence of sub von Neumann algebras of $\mathcal{A}$ with conditional expectations. For every $a \in \mathcal{A}$, the sequence $E^{\mathcal{A}_n}a$ converges ultrastrongly to $E^{\mathcal{A}_\infty}a$ as $n \to \infty$, where $\mathcal{A}_\infty = \bigvee_{n \geqq 1} \mathcal{A}_n$ (resp. $\mathcal{A}_\infty = \bigcap_{n \geqq 1} \mathcal{A}_n$).*

PROOF. As $\tilde{E}_n$ is an orthogonal projection on the Hilbert space $\mathfrak{H}$ and it is easy to see that any increasing orthogonal projections on a Hilbert space converges strongly.

Our proposition follows immediately. q.e.d.

THEOREM 4. *Let $\{\mathcal{A}_n\}$ be a decreasing sequence of sub von Neumann algebras of $\mathcal{A}$ with conditional expectations. For every $a \in \mathcal{A}$, the sequence $E^{\mathcal{A}_n}a$ converges ρ-almost uniformly to $E^{\mathcal{A}_\infty}a$ as $n \to \infty$.*

PROOF. Let $a \in \mathcal{A}$ and we define the kernel T as in Lemma 2. By Lance's ergodic theorem (cf. Lemma 1, clause 3) there exists $a_T \in \mathcal{A}$ such that $(1/q)\Sigma_{k<q}T^k a$ converges ρ-almost uniformly and ultrastrongly to a_T as $q \to \infty$. But we have

$$E_n a - a_T = \left(E_n a - \frac{1}{q_n} \sum_{k<q_n} T^k a \right) + \left(\frac{1}{q_n} \sum_{k<q_n} T^k a - a_T \right).$$

When $n \to \infty$, the norm of the first term of the RHS converges to zero by Lemma 2, the second term converges ρ-almost uniformly and ultrastrongly to zero, therefore $E_n a \to a_T$ ρ-almost uniformly and ultrastrongly, but Proposition 3 allows us to identify $a_T = E_\infty a$. q.e.d.

III. Convergence theorem for martingales

We begin with a technical result:

LEMMA 5. *Let $\mathcal{A}_1 \subset \cdots \subset \mathcal{A}_q$ be sub von Neumann algebras of $\mathcal{A}$ with conditional expectations $E^{\mathcal{A}_1}, \cdots, E^{\mathcal{A}_q}$. For every $a \in \mathcal{A}_+$, every $\varepsilon > 0$ we can write*

$$E^{\mathcal{A}_n}a = \eta_{n,q} + \eta'_{n,q}, \qquad \forall n \leqq q,$$

where $\eta_{n,q} \in \mathcal{A}$, $\eta'_{n,q} \in \mathcal{A}_+$ *and*

$$\|\eta_{n,q}\| \leqq \varepsilon,$$

$$\eta'_{n,q} \leqq c_q,$$

$\forall n \leqq q$ *with* $c_q \in \mathcal{A}_+$, $\|c_q\| \leqq 2\|a\|$ *and* $\rho(c_q) \leqq 4\|a\|^{1/2}\rho(a)^{1/2}$.

PROOF. Let

$$E^{\mathcal{A}_n}a = \left(E^{\mathcal{A}_n}a - \frac{1}{r_n}\sum_{0\leqq k<r_n} T_q^k a\right) + \frac{1}{r_n}\sum_{0\leqq k<r_n} T_q^k a,$$

where T_q is chosen as in Lemma 2 (in relation with the decreasing sequence $\{\mathcal{A}'_n\}$ defined by $\mathcal{A}'_n = \mathcal{A}_{q-n+1}$ if $n \leqq q-1$ and $\mathcal{A}_n = \mathcal{A}_1$ if $n \geqq q$) such that

$$\left\| E^{\mathcal{A}_n}a - \frac{1}{r_n}\sum_{0\leqq k<r_n} T_q^k a\right\| \leqq \varepsilon.$$

Let $\eta_{n,q}$ and $\eta'_{n,q}$ be the first and the second term of the above decomposition of $E^{\mathcal{A}_n}a$, the existence and properties of c_q follow from Lemma 1 clause 3.

LEMMA 6 (Maximal Lemma). *Let* $\{\mathcal{A}_n\}$ *be an increasing sequence of sub von Neumann algebras of* $\mathcal{A}$ *with conditional expectations.*

For every $a \in \mathcal{A}_+$, *there exists* $c \in \mathcal{A}_+$ *such that*

$$\|c\| \leqq 2\|a\|, \quad \rho(c) \leqq 4\|a\|^{1/2}\rho(a)^{1/2} \quad \textit{and} \quad E^{\mathcal{A}_n}a \leqq c,$$

for every $n \geqq 1$.

PROOF. For every integer $q \geqq 1$, let $\varepsilon_q = 1/2^q$ and consider the decomposition

$$E^{\mathcal{A}_n}a = \eta_{n,q} + \eta'_{n,q}, \qquad \forall n \leqq q$$

as in Lemma 5 with the associated $c_q \in \mathcal{A}_+$.

Let $c \in \mathcal{A}_+$ be an ultraweak limit of a sequence $\{c_{q_i}\}$; we have

$$\|c\| \leqq 2\|a\| \quad \text{and} \quad \rho(c) \leqq 4\|a\|^{1/2}\rho(a)^{1/2}.$$

As $\|\eta_{n,q_i}\| \leqq 1/2^{q_i} \xrightarrow[i]{} 0$, $\forall n \geqq 1$, we have

$$\|E^n a - \eta'_{n,q_i}\| \xrightarrow[i]{} 0, \qquad \forall n \geqq 1.$$

Taking the limit of the inequality:

$$\eta'_{n,q_i} \leqq c_{q_i}, \qquad \forall n \leqq q_i, \quad \forall i,$$

we obtain

$$E^{\mathscr{A}_n} a \leqq c, \qquad \forall n \geqq 1.$$

LEMMA 7. *Let $\{\mathscr{A}_n\}$ be an increasing sequence of sub von Neumann algebras with conditional expectations; let*

$$\mathscr{U}_0 = \{a \in \mathscr{A} \mid \lim_n \| E^{\mathscr{A}_n} a - E^{\mathscr{A}_\infty} a \| = 0\}.$$

Then every element a of $\mathscr{A}$ is an ultrastrong limit of a bounded sequence of $\mathscr{U}_0$.

PROOF. We remark first that $\bigcup_{n \geqq 1} \mathscr{A}_n \subset \mathscr{U}_0$ so that every element b of $\mathscr{A}_\infty$ is an ultrastrong limit of a bounded sequence b_p of $\mathscr{U}_0 \cap \mathscr{A}_\infty$.

Now let $a \in \mathscr{A}$, $b = E^{\mathscr{A}_\infty} a \in \mathscr{A}_\infty$ and the b_p's as above and let $a_p = a + (b_p - E^{\mathscr{A}_\infty} a)$; we have $E^{\mathscr{A}_n} a_p = E^{\mathscr{A}_n} b_p$, $\forall n \geqq 1$, $\forall p \geqq 1$. Therefore $a_p \in \mathscr{U}_0$, $\forall p \geqq 1$. It is clear that $a_p \to a$ ultrastrongly and $\sup_{p \geqq 1} \| a_p \| \leqq 2 \| a \| + \sup \| b_p \| < \infty$. q.e.d.

THEOREM 8. *Let $\{\mathscr{A}_n\}$ be an increasing sequence of sub von Neumann algebras of $\mathscr{A}$ with conditional expectations. For every $a \in \mathscr{A}$, the sequence $E^{\mathscr{A}_n} a$ converges ρ-almost uniformly to $E^{\mathscr{A}_\infty} a$ as $n \to \infty$, where $\mathscr{A}_\infty = \bigvee_{n \geqq 1} \mathscr{A}_n$.*

PROOF. By Lemma 1 clause 1, we can assume that $a \in \mathscr{A}_+$, $\| a \| \leqq 1$. By Lemma 7, there exists a sequence $a_p \in \mathscr{U}_0$, $a_p \to a$ ultrastrongly and $\| a_p \| \leqq 3$.

We have

$$E_n a - E_\infty a = E_n(a - a_p) + (E_n a_p - E_\infty a_p) + E_\infty(a_p - a). \tag{1}$$

Let $(a - a_p) = \sum_{j=1}^4 i^{j-1}(a - a_p)_{(j)}$ be the canonical decomposition of $(a - a_p)$ as in the introduction. Applying Lemma 6 to $(a - a_p)_{(j)}$ we can find ${}_jc_p \in \mathscr{A}_+$ such that

$$E_n((a - a_p)_{(j)}) \leqq {}_jc_p, \qquad n \geqq 1, \tag{2}$$

$$\| {}_jc_p \| \leqq 4, \quad \rho({}_jc_p) \leqq 8((a - a_p)^{++}), \quad j = 1,2,3,4.$$

Let

$$E_\infty(a - a_p) = \sum_{j=1}^4 i^{j-1} E_\infty((a_p - a)_{(j)}) \tag{3}$$

and

$$c_p = \sum_{j=1}^{4} ({}_jc_p + E_\infty((a_p - a)_{(j)})). \tag{4}$$

As c_p converges strongly to zero as $p \to \infty$, for every $\varepsilon > 0$, there exist, by Pedersen–Saito theorem (cf. [6], [7]), a subsequence p_k and a projection e of $\mathcal{A}$ such that $\rho(e) \geqq 1 - \varepsilon$ and $\lim_k \| c_{p_k} e \| = 0$.

By Lemma 1 clause 1 and relations (2), (4) we have

$$\lim_{k\to\infty} \sup_{n \geqq 1} \| E_n((a - a_{p_k})_{(j)})e \| = 0, \qquad j = 1,2,3,4, \tag{5}$$

$$\lim_{k\to\infty} \| E_\infty((a_{p_k} - a)_{(j)})e \| = 0, \qquad j = 1,2,3,4. \tag{6}$$

We have, by (1),

$$\|(E_n a - E_\infty a)e\| \leqq \| E_n a_{p_k} - E_\infty a_{p_k} \| + \sum_{j=1}^{4} (\| E_n((a - a_{p_k})_{(j)})e \| + \| E_\infty((a - a_{p_k}))e \|).$$

The second term of the RHS tends to zero as $k \to \infty$, *uniformly with respect to* n; it follows immediately, by the choice of the a_{p_k}'s, that $\lim_{n\to\infty} \|(E_n a - E_\infty a)e\| = 0$. q.e.d.

Let $\{\mathcal{A}_n\}$ be a fixed increasing sequence of sub von Neumann algebras of $\mathcal{A}$ with conditional expectations. We say that a sequence $\{a_n\}$ of elements of $\mathcal{A}$ is a *martingale* adapted to the sequence $\{\mathcal{A}_n\}$ if:

$$a_n \in \mathcal{A}_n, \qquad \forall n \geqq 1,$$

$$E^{\mathcal{A}_n} a_{n+1} = a_n, \qquad \forall n \geqq 1,$$

$$\sup_{n \geqq 1} \| a_n \| < \infty.$$

THEOREM 9. *Let $\{a_n\}$ be a martingale adapted to an increasing sequence $\{\mathcal{A}_n\}$ of sub von Neumann algebras of $\mathcal{A}$ with conditional expectations. There exists a unique $a \in \mathcal{A}_\infty = \bigvee_{n \geqq 1} \mathcal{A}_n$ such that $\{a_n\}$ converges to a ultrastrongly and ρ-almost uniformly. Furthermore $a_n = E^{\mathcal{A}_n} a$, $\forall n \geqq 1$.*

PROOF. The vectors $a_1\xi, (a_2 - a_1)\xi, \cdots, (a_{n+1} - a_n)\xi, \cdots$ are orthogonal since $\{\tilde{E}_n\}$ is an increasing sequence of orthogonal projections in $\mathfrak{H}$. Let $K = \sup_{n \geqq 1} \| a_n \| < \infty$; we have

$$a_n\xi = a_1\xi + \sum_{k=2}^{n} (a_k - a_{k-1})\xi,$$

$$\| a_n\xi \|^2 = \| a_1\xi \|^2 + \sum_{k=2}^{n} \| (a_k - a_{k-1})\xi \|^2 \leqq K^2, \qquad \forall n \geqq 1.$$

This implies that the series $a_1\xi + (a_2 - a_1)\xi + \cdots$ converges to a limit, say ψ, in $\mathfrak{H}$. It is clear that $\tilde{E}_n\psi = a_n\xi$, $\forall n \geqq 1$.

Let $a' \in \mathcal{A}'$; we have

$$\| a_n a'\xi - a'\psi \| = \| a'(a_n\xi - \psi) \| \leqq \| a' \| \cdot \| a_n\xi - \psi \| \xrightarrow[n\to\infty]{} 0.$$

As $\mathcal{A}'\xi$ is strongly dense in $\mathfrak{H}$ and $\sup_n \| a_n \| < \infty$, the sequence a_n converges ultrastrongly to a limit $a \in \mathcal{A}_\infty$ satisfying $a\xi = \psi$, as $\tilde{E}_n\psi = a_n\xi$, we have $a_n = E_n a$, $\forall n \geqq 1$. The theorem follows then from Theorem 8. q.e.d.

Added in proof. After the submission of this paper, we received a manuscript of E. C. Lance proving the convergence of martingales in the particular case of semi-finite von Neumann algebras.

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